

THE BOUNDEDNESS OF CERTAIN SUBLINEAR OPERATORS WITH ROUGH KERNEL GENERATED BY CALDERÓN-ZYGMUND OPERATORS AND THEIR COMMUTATORS ON GENERALIZED WEIGHTED MORREY SPACES

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ABSTRACT. The aim of this paper is to get the boundedness of certain sublinear operators with rough kernel generated by Calderón-Zygmund operators on the generalized weighted Morrey spaces under generic size conditions which are satisfied by most of the operators in harmonic analysis. We also prove that the commutator operators formed by *BMO* functions and certain sublinear operators with rough kernel are also bounded on the generalized weighted Morrey spaces. Marcinkiewicz operator which satisfies the conditions of these theorems can be considered as an example.

1. INTRODUCTION

The classical Morrey spaces $M_{p,\lambda}$ have been introduced by Morrey in [27] to study the local behavior of solutions of second order elliptic partial differential equations (PDEs). In recent years there has been an explosion of interest in the study of the boundedness of operators on Morrey-type spaces. It has been obtained that many properties of solutions to PDEs are concerned with the boundedness of some operators on Morrey-type spaces. In fact, better inclusion between Morrey and Hölder spaces allows to obtain higher regularity of the solutions to different elliptic and parabolic boundary problems (see [12, 31, 35, 36] for details).

Let \mathbb{R}^n be the n -dimensional Euclidean space of points $x = (x_1, \dots, x_n)$ with norm $|x| = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}$. Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B . For a given measurable set E , we also denote the Lebesgue measure of E by $|E|$. For any given $\Omega \subseteq \mathbb{R}^n$ and $0 < p < \infty$, denote by $L_p(\Omega)$ the spaces of all functions f satisfying

$$\|f\|_{L_p(\Omega)} = \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

We recall the definition of classical Morrey spaces $M_{p,\lambda}$ as

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$$M_{p,\lambda}(\mathbb{R}^n) = \left\{ f : \|f\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\},$$

where $f \in L_p^{loc}(\mathbb{R}^n)$, $0 \leq \lambda \leq n$ and $1 \leq p < \infty$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on \mathbb{R}^n . It is known that $M_{p,\lambda}(\mathbb{R}^n)$ is an expansion of $L_p(\mathbb{R}^n)$ in the sense that $L_{p,0} = L_p(\mathbb{R}^n)$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} \equiv \|f\|_{WM_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p -space of measurable functions f for which

$$\begin{aligned} \|f\|_{WL_p(B(x,r))} &\equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} \\ &= \sup_{t>0} t |\{y \in B(x,r) : |f(y)| > t\}|^{1/p} \\ &= \sup_{0 < t \leq |B(x,r)|} t^{1/p} (f\chi_{B(x,r)})^*(t) < \infty, \end{aligned}$$

where g^* denotes the non-increasing rearrangement of a function g .

Throughout the paper we assume that $x \in \mathbb{R}^n$ and $r > 0$ and also let $B(x,r)$ denotes the open ball centered at x of radius r , $B^C(x,r)$ denotes its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$.

Morrey has investigated that many properties of solutions to PDEs can be attributed to the boundedness of some operators on Morrey spaces. For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and the Calderón–Zygmund singular integral operator on these spaces, we refer the readers to [1, 3, 32]. For the properties and applications of classical Morrey spaces, see [4, 5, 11, 12] and references therein.

After studying Morrey spaces in detail, researchers have passed to generalized Morrey spaces. Mizuhara [26] has given generalized Morrey spaces $M_{p,\varphi}$ considering $\varphi(r)$ instead of r^λ in the above definition of the Morrey space. Later, Guliyev [15] and Karaman [20] have defined the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm as follows:

Definition 1. (Generalized Morrey space) Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \leq p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x, r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi} \big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \quad WM_{p,\lambda} = WM_{p,\varphi} \big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

During the last decades various classical operators, such as maximal, singular and potential operators have been widely investigated in classical and generalized Morrey spaces (see [2, 17, 20, 24] for details).

Maximal functions and singular integrals play a key role in harmonic analysis since maximal functions could control crucial quantitative information concerning the given functions, despite their larger size, while singular integrals, Hilbert transform as it's prototype, recently intimately connected with PDEs, operator theory and other fields.

Let $f \in L^{loc}(\mathbb{R}^n)$. The Hardy-Littlewood(H-L) maximal operator M is defined by

$$Mf(x) = \sup_{t>0} |B(x, t)|^{-1} \int_{B(x, t)} |f(y)| dy.$$

Let \bar{T} be a standard Calderón-Zygmund(C-Z) singular integral operator, briefly a C-Z operator, i.e., a linear operator bounded from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to the functions $f \in L_1^{loc}(\mathbb{R}^n)$ represented by

$$\bar{T}f(x) = p.v. \int_{\mathbb{R}^n} k(x-y)f(y) dy \quad x \notin \text{supp}f.$$

Such operators have been introduced in [8]. Here k is a C-Z kernel [14]. Chiarenza and Frasca [3] have obtained the boundedness of H-L maximal operator M and C-Z operator \bar{T} on $M_{p,\lambda}(\mathbb{R}^n)$. It is also well known that H-L maximal operator M and C-Z operator \bar{T} play an important role in harmonic analysis (see [13, 25, 39, 40, 41]). Also, the theory of the C-Z operator is one of the important achievements of classical analysis in the last century, which has many important applications in Fourier analysis, complex analysis, operator theory and so on.

Suppose that S^{n-1} is the unit sphere in \mathbb{R}^n ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ be homogeneous of degree zero. We define $s' = \frac{s}{s-1}$ for any $s > 1$. Suppose that T_Ω represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp}f$

$$(1.1) \quad |T_\Omega f(x)| \leq c_0 \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy,$$

where c_0 is independent of f and x .

For a locally integrable function b on \mathbb{R}^n , suppose that the commutator operator $T_{\Omega,b}$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp}f$

$$(1.2) \quad |T_{\Omega,b}f(x)| \leq c_0 \int_{\mathbb{R}^n} |b(x) - b(y)| \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy,$$

where c_0 is independent of f and x .

We point out that the condition (1.1) in the case $\Omega \equiv 1$ was first introduced by Soria and Weiss in [37]. The conditions (1.1) and (1.2) are satisfied by many interesting operators in harmonic analysis, such as the C-Z operators, Carleson's maximal operator, H-L maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory singular integrals, the Bochner-Riesz means and so on (see [23], [37] for details).

Let $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ be homogeneous of degree zero and satisfies the cancellation condition

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0,$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$. The C-Z singular integral operator with rough kernel \overline{T}_Ω is defined by

$$\overline{T}_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy,$$

satisfies the condition (1.1).

It is obvious that when $\Omega \equiv 1$, \overline{T}_Ω is the C-Z operator \overline{T} .

In 1976, Coifman et al. [6] introduced the commutator generated by \overline{T}_Ω and a local integrable function b as follows:

$$(1.3) \quad [b, \overline{T}_\Omega]f(x) \equiv b(x)\overline{T}_\Omega f(x) - \overline{T}_\Omega(bf)(x) = p.v. \int_{\mathbb{R}^n} [b(x) - b(y)] \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.$$

Sometimes, the commutator defined by (1.3) is also called the commutator in Coifman-Rocherberg-Weiss's sense, which has its root in the complex analysis and harmonic analysis (see [6]).

Remark 1. [35] *As another extension of Hilbert transform, a variety of operators related to the singular integrals for C-Z with homogeneous kernel, but lacking the smoothness required in the classical theory, have been studied. In this case, when Ω satisfies some size conditions, the kernel of the operator has no regularity, and so the operator is called rough integral operator. The theory of Operators with homogeneous kernel is a well studied area (see [14] and [25] for example). Lu et al. ([24]), Gurbuz et al. ([2]) and Gurbuz ([17]) have studied certain sublinear operators mentioned above with rough kernel on the generalized Morrey spaces. These include the operator $[b, \overline{T}_\Omega]$. For more results, we refer the reader to [2, 17, 18, 24, 25].*

In [2, 17], the boundedness of the sublinear operators with rough kernel generated by C-Z operators and their commutators on generalized Morrey spaces has been investigated.

In this paper, we first prove the boundedness of the sublinear operators with rough kernels T_Ω satisfying condition (1.1) generated by C-Z singular integral operators with rough kernel from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$ with the weight function w belonging to Muckenhoupt's class A_p for $1 < p < \infty$, and from the space $M_{1,\varphi_1}(w)$ to the weak space $WM_{1,\varphi_2}(w)$. Then, we also obtain the boundedness of the sublinear commutator operators $T_{\Omega,b}$ satisfying condition (1.2) generated by a C-Z type operator with rough kernel and b from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$ for $1 < p < \infty$, $b \in BMO$ (bounded mean oscillation). Provided that $b \in BMO$ and

$T_{\Omega,b}$ is a sublinear operator, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the commutator operators $T_{\Omega,b}$ from $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$ for $1 < p < \infty$. In all the cases the conditions for the boundedness of T_{Ω} and $T_{\Omega,b}$ are given in terms of Zygmund-type integral inequalities on (φ_1, φ_2) which do not assume any assumption on monotonicity of φ_1, φ_2 in r . Finally, as an example to the conditions of these theorems are satisfied, we can consider the Marcinkiewicz operator.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. We will also denote the conjugate exponent of $p > 1$ by $p' = \frac{p}{p-1}$ and $s > 1$ by $s' = \frac{s}{s-1}$.

2. WEIGHTED MORREY SPACES

A weight function is a locally integrable function on \mathbb{R}^n which takes values in $(0, \infty)$ almost everywhere. For a weight function w and a measurable set E , we define $w(E) = \int_E w(x)dx$, the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E . Given a weight function w , we say that w satisfies the doubling condition if there exists a constant $D > 0$ such that for any ball B , we have $w(2B) \leq Dw(B)$. When w satisfies this condition, we denote $w \in \Delta_2$, for short.

If w is a weight function, we denote by $L_p(w) \equiv L_p(\mathbb{R}^n, w)$ the weighted Lebesgue space defined by the norm

$$\|f\|_{L_p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty, \quad \text{when } 1 \leq p < \infty$$

and by $\|f\|_{L_{\infty,w}} = \operatorname{esssup}_{x \in \mathbb{R}^n} |f(x)|w(x)$ when $p = \infty$.

We denote by $WL_p(w)$ the weighted weak space consisting of all measurable functions f such that

$$\|f\|_{WL_p(w)} = \sup_{t>0} tw(\{x \in \mathbb{R}^n : |f(x)| > t\})^{\frac{1}{p}} < \infty.$$

We recall that a weight function w is in the Muckenhoupt's class $A_p(\mathbb{R}^n)$, $1 < p < \infty$, if

$$\begin{aligned} [w]_{A_p} &:= \sup_B [w]_{A_p(B)} \\ (2.1) \quad &= \sup_B \left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1} < \infty, \end{aligned}$$

where the supremum is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. The expression $[w]_{A_p}$ is called characteristic constant of w . Note that, for all balls B we have

$$(2.2) \quad [w]_{A_p}^{1/p} \geq [w]_{A_p(B)}^{1/p} = |B|^{-1} \|w\|_{L_1(B)}^{1/p} \|w^{-1/p}\|_{L_{p'}(B)} \geq 1$$

by the Hölder's inequality. For $p = 1$, the class $A_1(\mathbb{R}^n)$ is defined by

$$(2.3) \quad \frac{1}{|B|} \int_B w(x) dx \leq C \inf_{x \in B} w(x)$$

for every ball $B \subset \mathbb{R}^n$. Thus, we have the condition $Mw(x) \leq Cw(x)$ with $[w]_{A_1} = \sup_{x \in \mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and also for $p = \infty$ we define $A_\infty = \bigcup_{1 \leq p < \infty} A_p$, $[w]_{A_\infty} = \inf_{1 \leq p < \infty} [w]_{A_p}$ and $[w]_{A_\infty} \leq [w]_{A_p}$.

One knows that $A_p \subset A_s$ if $1 \leq p < s < \infty$, and that $w \in A_p$ for some $1 < p < s$ if $w \in A_s$ with $s > 1$, and also $[w]_{A_p} \leq [w]_{A_s}$.

By (2.1), we have

$$(2.4) \quad \left(w^{-\frac{p'}{p}}(B) \right)^{\frac{1}{p'}} = \left\| w^{-\frac{1}{p}} \right\|_{L_{p'}(B)} \leq C |B| w(B)^{-\frac{1}{p}}$$

for $1 < p < \infty$. Note that

$$(2.5) \quad \left(\operatorname{essinf}_{x \in E} f(x) \right)^{-1} = \operatorname{esssup}_{x \in E} \frac{1}{f(x)}$$

is true for any real-valued nonnegative function f and is measurable on E (see [42] page 143) and (2.3); we get

$$(2.6) \quad \begin{aligned} \|w^{-1}\|_{L_\infty(B)} &= \operatorname{esssup}_{x \in B} \frac{1}{w(x)} \\ &= \frac{1}{\operatorname{essinf}_{x \in B} w(x)} \leq C |B| w(B)^{-1}. \end{aligned}$$

Proposition 1. (see [30]) *Since definition of the Muckenhoupt's class $A_p(\mathbb{R}^n)$, we have*

$$w^{1-p'} \in A_{\frac{p'}{s'}} \text{ implies } [w^{1-p'}]_{A_{\frac{p'}{s'}}(B)}^{\frac{s'}{p'}} = |B|^{-1} \|w^{1-p'}\|_{L_1(B)}^{\frac{s'}{p'}} \|w^{\frac{s'}{p}}\|_{L_{(\frac{p'}{s'})'}(B)}$$

for $1 < p < \infty$. Since $w^{1-p'} \in A_{\frac{p'}{s'}} \subset A_{p'}$, we also know $w^{1-p'} \in A_{\frac{p'}{s'}}$ implies $w^{1-p'} \in A_{p'}$. Thus, we have

$$(2.7) \quad [w^{1-p'}]_{A_{p'}(B)}^{\frac{1}{p'}} = |B|^{-1} \|w^{1-p'}\|_{L_1(B)}^{\frac{1}{p'}} \|w^{\frac{1}{p}}\|_{L_p(B)}.$$

But, the converse of this implication is not generally valid.

Proposition 2. (see [30]) *To make the proofs simpler, we can also write $w^{1-p'} \in A_{\frac{p'}{s'}}$ as follows:*

$$(2.8) \quad \begin{aligned} [w^{1-p'}]_{A_{\frac{p'}{s'}}(B)}^{\frac{s(p-1)}{p(s-1)}} &= |B|^{-1} \|w^{1-p'}\|_{L_1(B)}^{\frac{s(p-1)}{p(s-1)}} \|w^{\frac{s'}{p}}\|_{L_{(\frac{p'}{s'})'}(B)} \\ [w^{1-p'}]_{A_{\frac{p'}{s'}}(B)}^{\frac{1}{p'}} &= |B|^{-\frac{s-1}{s}} \|w^{1-p'}\|_{L_1(B)}^{\frac{1}{p'}} \|w\|_{L_{\frac{s}{s-p}}(B)}^{\frac{1}{p}}, \end{aligned}$$

where

$$1 - p' = -\frac{p'}{p}, \quad \frac{s'}{p} = \frac{s}{p(s-1)}, \quad \frac{s'}{p'} = \frac{s(p-1)}{p(s-1)}, \quad \left(\frac{s}{p}\right)' = \frac{s}{s-p}, \quad \left(\frac{p'}{s'}\right)' = \frac{p(s-1)}{s-p}.$$

In the equation (2.8) if we write (2.7) instead of $\|w^{1-p'}\|_{L_1(B)}^{\frac{1}{p'}}$, then we obtain

$$(2.9) \quad [w^{1-p'}]_{A_{\frac{p'}{s}}(B)}^{\frac{1}{p'}} = |B|^{\frac{1}{s}} [w^{1-p'}]_{A_{p'}(B)}^{\frac{1}{p'}} \|w^{\frac{1}{p}}\|_{L_p(B)}^{-1} \|w\|_{L_{\frac{s}{s-p}}(B)}^{\frac{1}{p}}.$$

Lemma 1. (see [30]) Let $1 < p < s$ and $w^{1-p'} \in A_{\frac{p'}{s}}$. Then, the inequality

$$\|\Omega(\cdot - y)\|_{L_{p,w}(B)} \lesssim \|\Omega(\cdot - y)\|_{L_s(B)} \|w\|_{L_{(\frac{s}{p})'}(B)}^{\frac{1}{p}}$$

holds for every $y \in \mathbb{R}^n$ and for any ball $B \subset \mathbb{R}^n$.

The classical $A_p(\mathbb{R}^n)$ weight theory has been introduced by Muckenhoupt in the study of weighted L_p -boundedness of H-L maximal function in [28].

It is known from [14] that

Lemma 2. The following statements hold.

(1) If $w \in A_p$ for some $1 \leq p < \infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$ we have

$$w(\lambda B) \leq \lambda^{np} [w]_{A_p} w(B).$$

(2) If $w \in A_\infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$ we have

$$w(\lambda B) \leq 2^{\lambda^n} [w]_{A_\infty}^{\lambda^n} w(B).$$

(3) If $w \in A_p$ for some $1 \leq p \leq \infty$, then there exist $C > 0$ and $\delta > 0$ such that for any ball B and a measurable set $S \subset B$,

$$\frac{1}{[w]_{A_p}} \left(\frac{|S|}{|B|} \right) \leq \frac{w(S)}{w(B)} \leq C \left(\frac{|S|}{|B|} \right)^\delta.$$

(4) The function $w^{-\frac{1}{p-1}}$ is in $A_{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p < \infty$ with characteristic constant

$$[w^{-\frac{1}{p-1}}]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}.$$

Komori and Shirai [21] have introduced a version of the weighted Morrey space $L_{p,\kappa}(w)$, which is a natural generalization of the weighted Lebesgue space $L_p(w)$, and have investigated the boundedness of classical operators in harmonic analysis.

Definition 2. (Weighted Morrey space) Let $1 \leq p < \infty$, $0 < \kappa < 1$ and w be a weight function. We denote by $L_{p,\kappa}(w) \equiv L_{p,\kappa}(\mathbb{R}^n, w)$ the weighted Morrey space of all classes of locally integrable functions f with the norm

$$\|f\|_{L_{p,\kappa}(w)} = \sup_{x \in \mathbb{R}^n, r > 0} w(B(x, r))^{-\frac{\kappa}{p}} \|f\|_{L_{p,w}(B(x, r))} < \infty.$$

Furthermore, by $WL_{p,\kappa}(w) \equiv WL_{p,\kappa}(\mathbb{R}^n, w)$ we denote the weak weighted Morrey space of all classes of locally integrable functions f with the norm

$$\|f\|_{WL_{p,\kappa}(w)} = \sup_{x \in \mathbb{R}^n, r > 0} w(B(x, r))^{-\frac{\kappa}{p}} \|f\|_{WL_{p,w}(B(x, r))} < \infty.$$

Remark 2. Alternatively, we could define the weighted Morrey spaces with cubes instead of balls. Hence we shall use these two definitions of weighted Morrey spaces appropriate to calculation.

Remark 3. (1) If $w \equiv 1$ and $\kappa = \lambda/n$ with $0 \leq \lambda \leq n$, then $L_{p,\lambda/n}(1) = M_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey spaces.

(2) If $\kappa = 0$, then $L_{p,0}(w) = L_p(w)$ is the weighted Lebesgue spaces.

The following theorem has been proved in [21].

Theorem 1. *Let $1 \leq p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. Then the operators M and \overline{T} are bounded on $L_{p,\kappa}(w)$ for $p > 1$ and from $L_{1,\kappa}(w)$ to $WL_{1,\kappa}(w)$.*

3. SUBLINEAR OPERATORS WITH ROUGH KERNEL GENERATED BY CALDERÓN-ZYGMUND OPERATORS ON THE GENERALIZED WEIGHTED MORREY SPACES $M_{p,\varphi}(w)$

The generalized weighted Morrey spaces $M_{p,\varphi}(w)$ have been introduced by Guliyev [16] and Karaman [20] as follows.

Definition 3. (Generalized weighted Morrey space) *Let $1 \leq p < \infty$, $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and w be non-negative measurable function on \mathbb{R}^n . We denote by $M_{p,\varphi}(w) \equiv M_{p,\varphi}(\mathbb{R}^n, w)$ the generalized weighted Morrey space, the space of all classes of functions $f \in L_{p,w}^{loc}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} w(B(x, r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x, r))},$$

where $L_{p,w}(B(x, r))$ denotes the weighted $L_{p,w}$ -space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x, r))} \equiv \|f\chi_{B(x, r)}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int_{B(x, r)} |f(y)|^p w(y) dy \right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w) \equiv WM_{p,\varphi}(\mathbb{R}^n, w)$ we denote the weak generalized weighted Morrey space of all classes of functions $f \in WL_{p,w}^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} w(B(x, r))^{-\frac{1}{p}} \|f\|_{WL_{p,w}(B(x, r))} < \infty,$$

where $WL_{p,w}(B(x, r))$ denotes the weighted weak $WL_{p,w}$ -space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x, r))} \equiv \|f\chi_{B(x, r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t > 0} tw(\{y \in B(x, r) : |f(y)| > t\})^{\frac{1}{p}} < \infty.$$

Remark 4. (1) If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space.

(2) If $\varphi(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$, $0 < \kappa < 1$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.

(3) If $\varphi(x, r) \equiv \nu(B(x, r))^{\frac{\kappa}{p}} w(B(x, r))^{-\frac{1}{p}}$, $0 < \kappa < 1$, then $M_{p,\varphi}(w) = L_{p,\kappa}(\nu, w)$ is the two weighted Morrey space.

(4) If $w \equiv 1$ and $\varphi(x, r) = r^{\frac{\lambda-n}{p}}$ with $0 \leq \lambda \leq n$, then $M_{p,\varphi}(1) = M_{p,\lambda}$ is the classical Morrey space and $WM_{p,\varphi}(1) = WM_{p,\lambda}$ is the weak Morrey space.

(5) If $\varphi(x, r) \equiv w(B(x, r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_p(w)$ is the weighted Lebesgue space.

Inspired by the above results, in this paper we are interested in the boundedness of sublinear operators with rough kernel on generalized weighted Morrey spaces and give bounded mean oscillation space estimates for their commutators.

In this section we prove boundedness of the operator T_Ω satisfying (1.1) on the generalized weighted Morrey spaces $M_{p,\varphi}(w)$ by using the following main Lemma 3.

Theorem 2. (see [25]) Let $\Omega \in L_s(S^{n-1})$, $s > 1$, be homogeneous of degree zero, and $1 \leq p < \infty$. If p , q and the weight function w satisfy one of the following statements:

- (i) $s' \leq p < \infty$, $p \neq 1$ and $w \in A_{\frac{p}{s'}};$
 - (ii) $1 < p \leq s$, $p \neq \infty$ and $w^{1-p'} \in A_{\frac{p'}{s'}};$
 - (iii) $1 < p < \infty$, and $w^{s'} \in A_p$,
- then \overline{T}_Ω is bounded on $L_p(w)$.

We first prove the following main Lemma 3.

Lemma 3. (Our main Lemma) Let $\Omega \in L_s(S^{n-1})$, $s > 1$, be homogeneous of degree zero, and $1 \leq p < \infty$. Let T_Ω be a sublinear operator satisfying condition (1.1), bounded on $L_p(w)$ for $p > 1$ and bounded from $L_1(w)$ to $WL_1(w)$.

If $p > 1$, $s' \leq p$ and $w \in A_{\frac{p}{s'}}$, then the inequality

$$(3.1) \quad \|T_\Omega f\|_{L_{p,w}(B(x_0,r))} \lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0, r)$ and for all $f \in L_{p,w}^{loc}(\mathbb{R}^n)$.

If $p > 1$, $p < s$ and $w^{1-p'} \in A_{\frac{p'}{s'}}$, then the inequality

$$\|T_\Omega f\|_{L_{p,w}(B(x_0,r))} \lesssim \|w\|_{L_{\frac{s}{s-p}}^{\frac{1}{p}}(B(x_0,r))} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w\|_{L_{\frac{s}{s-p}}^{-\frac{1}{p}}(B(x_0,t))} \frac{dt}{t}$$

holds for any ball $B(x_0, r)$ and for all $f \in L_{p,w}^{loc}(\mathbb{R}^n)$.

Moreover, for $s > 1$ the inequality

$$(3.2) \quad \|T_\Omega f\|_{WL_{1,w}(B(x_0,r))} \lesssim w(B(x_0,r)) \int_{2r}^{\infty} \|f\|_{L_{1,w}(B(x_0,t))} w(B(x_0,t))^{-1} \frac{dt}{t}$$

holds for any ball $B(x_0, r)$ and for all $f \in L_{1,w}^{loc}(\mathbb{R}^n)$.

Proof. For $x \in B(x_0, t)$, notice that Ω is homogenous of degree zero and $\Omega \in L_s(S^{n-1})$, $s > 1$. Then, we obtain

$$\begin{aligned}
 \left(\int_{B(x_0, t)} |\Omega(x-y)|^s dy \right)^{\frac{1}{s}} &= \left(\int_{B(x-x_0, t)} |\Omega(z)|^s dz \right)^{\frac{1}{s}} \\
 &\leq \left(\int_{B(0, t+|x-x_0|)} |\Omega(z)|^s dz \right)^{\frac{1}{s}} \\
 &\leq \left(\int_{B(0, 2t)} |\Omega(z)|^s dz \right)^{\frac{1}{s}} \\
 &= \left(\int_0^{2t} \int_{S^{n-1}} |\Omega(z')|^s d\sigma(z') r^{n-1} dr \right)^{\frac{1}{s}} \\
 (3.3) \quad &= C \|\Omega\|_{L_s(S^{n-1})} |B(x_0, 2t)|^{\frac{1}{s}}.
 \end{aligned}$$

Let $1 < p < \infty$, $s' \leq p$ and $w \in A_{\frac{p}{s'}}^p$. For any $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r and $2B = B(x_0, 2r)$. We represent f as

$$(3.4) \quad f = f_1 + f_2, \quad f_1(y) = f(y) \chi_{2B}(y), \quad f_2(y) = f(y) \chi_{(2B)^c}(y), \quad r > 0$$

and have

$$\|T_\Omega f\|_{L_{p,w}(B)} \leq \|T_\Omega f_1\|_{L_{p,w}(B)} + \|T_\Omega f_2\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_p(w)$, $T_\Omega f_1 \in L_p(w)$ and by the boundedness of T_Ω on $L_p(w)$ (see Theorem 2) it follows that:

$$\|T_\Omega f_1\|_{L_{p,w}(B)} \leq \|T_\Omega f_1\|_{L_{p,w}(\mathbb{R}^n)} \leq C \|f_1\|_{L_{p,w}(\mathbb{R}^n)} = C \|f\|_{L_{p,w}(2B)},$$

where constant $C > 0$ is independent of f .

It is clear that $x \in B$, $y \in (2B)^c$ implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. We get

$$|T_\Omega f_2(x)| \leq 2^n c_1 \int_{(2B)^c} \frac{|f(y)| |\Omega(x-y)|}{|x_0 - y|^n} dy.$$

By the Fubini's theorem, we have

$$\begin{aligned}
 \int_{(2B)^c} \frac{|f(y)| |\Omega(x-y)|}{|x_0 - y|^n} dy &\approx \int_{(2B)^c} |f(y)| |\Omega(x-y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\
 &\approx \int_{2r}^{\infty} \int_{2r \leq |x_0-y| \leq t} |f(y)| |\Omega(x-y)| dy \frac{dt}{t^{n+1}} \\
 (3.5) \quad &\lesssim \int_{2r}^{\infty} \int_{B(x_0, t)} |f(y)| |\Omega(x-y)| dy \frac{dt}{t^{n+1}}.
 \end{aligned}$$

Applying the Hölder's inequality and by (3.3) and (2.4), we get

$$\begin{aligned}
& \int_{(2B)^C} \frac{|f(y)| |\Omega(x-y)|}{|x_0-y|^n} dy \\
& \lesssim \int_{2r}^{\infty} \|\Omega(x-\cdot)\|_{L_s(B(x_0,t))} \|f\|_{L_{s'}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
& \lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \left\| w^{-\frac{s'}{p}} \right\|_{L_{(\frac{p}{s'})'}(B(x_0,t))}^{\frac{1}{s'}} |B(x_0, 2t)|^{\frac{1}{s}} \frac{dt}{t^{n+1}} \\
& \lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0, t))^{-\frac{1}{p}} |B(x_0, t)|^{\frac{1}{s'}} |B(x_0, 2t)|^{\frac{1}{s}} \frac{dt}{t^{n+1}} \\
(3.6) \quad & \lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0, t))^{-\frac{1}{p}} \frac{dt}{t}.
\end{aligned}$$

Thus, by (3.6), it follows that:

$$|T_{\Omega} f_2(x)| \lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0, t))^{-\frac{1}{p}} \frac{dt}{t}.$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$(3.7) \quad \|T_{\Omega} f_2\|_{L_{p,w}(B)} \lesssim w(B(x_0, r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0, t))^{-\frac{1}{p}} \frac{dt}{t}$$

is valid. Thus,

$$\|T_{\Omega} f\|_{L_{p,w}(B)} \lesssim \|f\|_{L_{p,w}(2B)} + w(B(x_0, r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0, t))^{-\frac{1}{p}} \frac{dt}{t}.$$

On the other hand, it is clear that $w \in A_{\frac{p}{s'}}$ implies $w \in A_p$, by (2.2) and (2.4) we have

$$\begin{aligned}
\|f\|_{L_{p,w}(2B)} &\approx |B| \|f\|_{L_{p,w}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\
&\lesssim |B| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim w(B(x_0,r))^{\frac{1}{p}} \left\| w^{-\frac{1}{p}} \right\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \left\| w^{-\frac{1}{p}} \right\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
(3.8) \quad &\lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}.
\end{aligned}$$

By combining the above inequalities, we obtain

$$\|T_{\Omega}f\|_{L_{p,w}(B(x_0,r))} \lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}.$$

Let $1 < p < s$ and $w^{1-p'} \in A_{\frac{p'}{s'}}$. Similarly to (3.3), when $y \in B(x_0, t)$, it is true that

$$(3.9) \quad \left(\int_{B(x_0,r)} |\Omega(x-y)|^s dy \right)^{\frac{1}{s}} \leq C \|\Omega\|_{L_s(S^{n-1})} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}}.$$

By the Fubini's theorem, the Minkowski inequality, Lemma 1, (3.9) and the Hölder's inequality, respectively we get

$$\begin{aligned}
\|T_\Omega f_2\|_{L_{p,w}(B)} &\leq \left(\int_B \left| \int_{2r}^\infty \int_{B(x_0,t)} |f(y)| |\Omega(x-y)| dy \frac{dt}{t^{n+1}} \right|^p w(x) dx \right)^{\frac{1}{p}} \\
&\leq \int_{2r}^\infty \int_{B(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_{p,w}(B)} dy \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^\infty \int_{B(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_s(B)} \|w\|_{L_{(\frac{s}{p})'}(B)}^{\frac{1}{p}} dy \frac{dt}{t^{n+1}} \\
&\lesssim \|w\|_{L_{(\frac{s}{p})'}(B)}^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_1(B(x_0,t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{n+1}} \\
&\lesssim \|w\|_{L_{\frac{s}{s-p}}(B(x_0,r))}^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} \left\| w^{-\frac{p'}{p}} \right\|_{L_1(B(x_0,t))}^{\frac{1}{p'}} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{n+1}} \\
&\lesssim |B(x_0, r)|^{\frac{1}{s}} \|w\|_{L_{\frac{s}{s-p}}(B(x_0,r))}^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} \left\| w^{1-p'} \right\|_{L_1(B(x_0,t))}^{\frac{1}{p'}} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{n+1}}.
\end{aligned}$$

Applying (2.7) and (2.9) for $\left\| w^{1-p'} \right\|_{L_1(B(x_0,t))}^{\frac{1}{p'}}$ and $\|w\|_{L_{\frac{s}{s-p}}(B(x_0,r))}^{\frac{1}{p}}$, respectively we have

$$\|T_\Omega f_2\|_{L_{p,w}(B(x_0,r))} \lesssim \|w\|_{L_{\frac{s}{s-p}}(B(x_0,r))}^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} \|w\|_{L_{\frac{s}{s-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}.$$

Therefore,

$$\|T_\Omega f\|_{L_{p,w}(B)} \lesssim \|f\|_{L_{p,w}(2B)} + \|w\|_{L_{\frac{s}{s-p}}(B(x_0,r))}^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} \|w\|_{L_{\frac{s}{s-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}.$$

On the other hand, we have

$$\begin{aligned}
\|f\|_{L_{p,w}(2B)} &\approx |B| \|f\|_{L_{p,w}(2B)} \int_{2r}^\infty \frac{dt}{t^{n+1}} \\
&\lesssim |B| \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim |B|^{\frac{1}{s}} \left\| w^{1-p'} \right\|_{L_1(B)}^{\frac{1}{p'}} \|w\|_{L_{\frac{s}{s-p}}(B)}^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim \|w\|_{L_{\frac{s}{s-p}}(B(x_0,r))}^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} \|w\|_{L_{\frac{s}{s-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}.
\end{aligned}$$

By combining the above inequalities, we obtain

$$\|T_\Omega f\|_{L_{p,w}(B(x_0,r))} \lesssim \|w\|_{L_{\frac{s}{s-p}}(B(x_0,r))}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w\|_{L_{\frac{s}{s-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}.$$

Let $p = 1 < s \leq \infty$. From the weak $(1, 1)$ boundedness of T_Ω and (3.8) it follows that:

$$\begin{aligned} \|T_\Omega f_1\|_{WL_{1,w}(B)} &\leq \|T_\Omega f_1\|_{WL_{1,w}(\mathbb{R}^n)} \lesssim \|f_1\|_{L_{1,w}(\mathbb{R}^n)} \\ (3.10) \quad &= \|f\|_{L_{1,w}(2B)} \lesssim w(B(x_0, r)) \int_{2r}^{\infty} \|f\|_{L_{1,w}(B(x_0,t))} w(B(x_0,t))^{-1} \frac{dt}{t}. \end{aligned}$$

Then from (3.7) and (3.10) we get the inequality (3.2), which completes the proof. \square

In the following theorem we get the boundedness of the operator T_Ω on the generalized weighted Morrey spaces $M_{p,\varphi}(w)$.

Theorem 3. (Our main result) Let $\Omega \in L_s(S^{n-1})$, $s > 1$, be homogeneous of degree zero, and $1 \leq p < \infty$. Let T_Ω be a sublinear operator satisfying condition (1.1), bounded on $L_p(w)$ for $p > 1$ and bounded from $L_1(w)$ to $WL_1(w)$. Let also, for $s' \leq p$, $p \neq 1$ and $w \in A_{\frac{p}{s'}}^p$, the pair (φ_1, φ_2) satisfies the condition

$$(3.11) \quad \int_r^\infty \frac{\operatorname{ess\,inf}_{t < \tau < \infty} \varphi_1(x, \tau) w(B(x, \tau))^{\frac{1}{p}}}{w(B(x, t))^{\frac{1}{p}}} \frac{dt}{t} \leq C \varphi_2(x, r),$$

and for $1 < p < s$ and $w^{1-p'} \in A_{\frac{p'}{s'}}^p$ the pair (φ_1, φ_2) satisfies the condition

$$(3.12) \quad \int_r^\infty \frac{\operatorname{ess\,inf}_{t < \tau < \infty} \varphi_1(x, \tau) \|w\|_{L_{\frac{s}{s-p}}(B(x,r))}^{\frac{1}{p}}}{\|w\|_{L_{\frac{s}{s-p}}(B(x,t))}^{\frac{1}{p}}} \frac{dt}{t} \leq C \varphi_2(x, r) \frac{w(B(x, r))^{\frac{1}{p}}}{\|w\|_{L_{\frac{s}{s-p}}(B(x,r))}^{\frac{1}{p}}},$$

where C does not depend on x and r .

Then the operator T_Ω is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$. Moreover, we have for $p > 1$

$$(3.13) \quad \|T_\Omega f\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)},$$

and for $p = 1$

$$(3.14) \quad \|T_\Omega f\|_{WM_{1,\varphi_2}(w)} \lesssim \|f\|_{M_{1,\varphi_1}(w)}.$$

Proof. since $f \in M_{p,\varphi_1}(w)$, by (2.5) and the non-decreasing, with respect to t , of the norm $\|f\|_{L_{p,w}(B(x_0,t))}$, we get

$$\begin{aligned} & \frac{\|f\|_{L_{p,w}(B(x_0,t))}}{\operatorname{ess\,inf}_{0 < t < \tau < \infty} \varphi_1(x_0, \tau) w(B(x_0, \tau))^{\frac{1}{p}}} \\ & \leq \operatorname{ess\,sup}_{0 < t < \tau < \infty} \frac{\|f\|_{L_{p,w}(B(x_0,t))}}{\varphi_1(x_0, \tau) w(B(x_0, \tau))^{\frac{1}{p}}} \\ & \leq \operatorname{ess\,sup}_{0 < \tau < \infty} \frac{\|f\|_{L_{p,w}(B(x_0,\tau))}}{\varphi_1(x_0, \tau) w(B(x_0, \tau))^{\frac{1}{p}}} \\ & \leq \|f\|_{M_{p,\varphi_1}(w)}. \end{aligned}$$

For $s' \leq p < \infty$, since (φ_1, φ_2) satisfies (3.11), we have

$$\begin{aligned} & \int_r^\infty \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t} \\ & \leq \int_r^\infty \frac{\|f\|_{L_{p,w}(B(x_0,t))}}{\operatorname{ess\,inf}_{t < \tau < \infty} \varphi_1(x_0, \tau) w(B(x_0, \tau))^{\frac{1}{p}}} \frac{\operatorname{ess\,inf}_{t < \tau < \infty} \varphi_1(x_0, \tau) w(B(x_0, \tau))^{\frac{1}{p}}}{w(B(x_0, t))^{\frac{1}{p}}} \frac{dt}{t} \\ & \leq C \|f\|_{M_{p,\varphi_1}(w)} \int_r^\infty \frac{\operatorname{ess\,inf}_{t < \tau < \infty} \varphi_1(x_0, \tau) w(B(x_0, \tau))^{\frac{1}{p}}}{w(B(x_0, t))^{\frac{1}{p}}} \frac{dt}{t} \\ & \leq C \|f\|_{M_{p,\varphi_1}(w)} \varphi_2(x_0, r). \end{aligned}$$

Then by (3.1), we get

$$\begin{aligned} \|T_\Omega f\|_{M_{p,\varphi_2}(w)} &= \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_2(x_0, r)^{-1} w(B(x_0, r))^{-\frac{1}{p}} \|T_\Omega f\|_{L_{p,w}(B(x_0, r))} \\ &\leq C \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_2(x_0, r)^{-1} \int_r^\infty \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0, t))^{-\frac{1}{p}} \frac{dt}{t} \\ &\leq C \|f\|_{M_{p,\varphi_1}(w)}. \end{aligned}$$

For the case of $1 \leq p < s$, we can also use the same method, so we omit the details. This completes the proof of Theorem 3. \square

Let $f \in L_1^{loc}(\mathbb{R}^n)$. The rough Hardy-Littlewood maximal operator M_Ω is defined by

$$M_\Omega f(x) = \sup_{t > 0} \frac{1}{|B(x, t)|} \int_{B(x,t)} |\Omega(x-y)| |f(y)| dy.$$

Then we can get the following corollary.

Corollary 1. *Let $1 \leq p < \infty$, $\Omega \in L_s(S^{n-1})$, $s > 1$, be homogeneous of degree zero. For $s' \leq p$, $p \neq 1$ and $w \in A_{\frac{p}{s'}}^p$, the pair (φ_1, φ_2) satisfies condition (3.11) and for $1 < p < s$ and $w^{1-p'} \in A_{\frac{p'}{s'}}^{p'}$ the pair (φ_1, φ_2) satisfies condition (3.12).*

Then the operators M_Ω and \overline{T}_Ω are bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.

In the case of $w = 1$ from Theorem 3, we get

Corollary 2. (see [2, 17]) Let $\Omega \in L_s(S^{n-1})$, $s > 1$, be homogeneous of degree zero, and $1 \leq p < \infty$. Let T_Ω be a sublinear operator satisfying condition (1.1), bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Let also, for $s' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies the condition

$$\int_r^\infty \frac{\text{essinf}_{t \leq \tau < \infty} \varphi_1(x, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \leq C \varphi_2(x, r),$$

and for $1 < p < s$ the pair (φ_1, φ_2) satisfies the condition

$$\int_r^\infty \frac{\text{essinf}_{t \leq \tau < \infty} \varphi_1(x, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{p} - \frac{n}{s} + 1}} dt \leq C \varphi_2(x, r) r^{\frac{n}{s}},$$

where C does not depend on x and r .

Then the operator T_Ω is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2} . Moreover, we have for $p > 1$

$$\|T_\Omega f\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}},$$

and for $p = 1$

$$\|T_\Omega f\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

In the case of $\varphi_1(x, r) = \varphi_2(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$ from Theorem 3 we get the following new result.

Corollary 3. Let $1 \leq p < \infty$, $\Omega \in L_s(S^{n-1})$, $s > 1$, be homogeneous of degree zero and $0 < \kappa < 1$. Let also T_Ω be a sublinear operator satisfying condition (1.1), bounded on $L_p(w)$ for $p > 1$ and bounded from $L_1(w)$ to $WL_1(w)$. For $s' \leq p$, $p \neq 1$ and $w \in A_{s'}^{\frac{p}{p-1}}$ or $1 < p < s$ and $w^{1-p'} \in A_{s'}^{\frac{p'}{p}}$, the operator T_Ω is bounded on the weighted Morrey spaces $L_{p,\kappa}(w)$ for $p > 1$ and bounded from $L_{1,\kappa}(w)$ to $WL_{1,\kappa}(w)$.

When $\Omega \equiv 1$, from Theorem 3 we get

Corollary 4. Let $1 \leq p < \infty$, $w \in A_p$ and the pair (φ_1, φ_2) satisfies condition (3.11). Let also T be a sublinear operator satisfying condition (1.1), bounded on $L_p(w)$ for $p > 1$ and bounded from $L_1(w)$ to $WL_1(w)$. Then the operator T is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.

Remark 5. Corollary 4 has been proved in [20].

When $\Omega \equiv 1$, In the case of $\varphi_1(x, r) = \varphi_2(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$ from Theorem 3 we get the following new result.

Corollary 5. $1 \leq p < \infty$, $0 < \kappa < 1$ and $w \in A_p$. Let also T be a sublinear operator satisfying condition (1.1), bounded on $L_p(w)$ for $p > 1$ and bounded from $L_1(w)$ to $WL_1(w)$. Then the operator T is bounded on the weighted Morrey spaces $L_{p,\kappa}(w)$ for $p > 1$ and bounded from $L_{1,\kappa}(w)$ to $WL_{1,\kappa}(w)$.

Remark 6. Note that, from Corollary 5, we get Theorem 1.

4. COMMUTATORS OF SUBLINEAR OPERATORS WITH ROUGH KERNEL
GENERATED BY CALDERÓN-ZYGMUND TYPE OPERATORS ON THE
GENERALIZED WEIGHTED MORREY SPACES $M_{p,\varphi}(w)$

In this section we prove the boundedness of the operator $T_{\Omega,b}$ satisfying condition (1.2) with $b \in BMO(\mathbb{R}^n)$ on the generalized weighted Morrey spaces $M_{p,\varphi}(w)$ by using the following main Lemma 7.

Let us recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 4. Suppose that $b \in L_1^{loc}(\mathbb{R}^n)$, let

$$\|b\|_* = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y) dy.$$

Define

$$BMO(\mathbb{R}^n) = \{b \in L_1^{loc}(\mathbb{R}^n) : \|b\|_* < \infty\}.$$

If one regards two functions whose difference is a constant as one, then the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to norm $\|\cdot\|_*$.

An early work about $BMO(\mathbb{R}^n)$ space can be attributed to John and Nirenberg [19]. For $1 < p < \infty$, there is a close relation between $BMO(\mathbb{R}^n)$ and A_p weights:

$$BMO(\mathbb{R}^n) = \{\alpha \log w : w \in A_p, \alpha \geq 0\}.$$

Let T be a linear operator. For a locally integrable function b on \mathbb{R}^n , we define the commutator $[b, T]$ by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x)$$

for any suitable function f . Since $L_\infty(\mathbb{R}^n) \subsetneq BMO(\mathbb{R}^n)$, the boundedness of $[b, T]$ is worse than T (e.g., the singularity; see also [33]). Therefore, many authors want to know whether $[b, T]$ shares the similar boundedness with T . There are a lot of articles that deal with the topic of commutators of different operators with BMO functions on Lebesgue spaces. The first results for this commutator have been obtained by Coifman et al. [6] in their study of certain factorization theorems for generalized Hardy spaces. Let \bar{T} be a C-Z operator. A well known result of Coifman et al. [6] states that when $K(x) = \frac{\Omega(x')}{|x|^n}$ and Ω is smooth, the commutator $[b, \bar{T}]f = b\bar{T}f - \bar{T}(bf)$ is bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$. The commutators of C-Z operator play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order (see, for example, [4, 5, 11, 34]). The boundedness of the commutator has been generalized to other contexts and important applications to some non-linear PDEs have been given by Coifman et al. [7].

The following lemmas about $BMO(\mathbb{R}^n)$ functions will help us to prove Lemma 7 and Theorem 5.

Lemma 4. (see [[29], Theorem 5, page 236]) Let $w \in A_\infty$. Then the norm of $BMO(w)$ is equivalent to the norm of $BMO(\mathbb{R}^n)$, where

$$BMO(w) = \{b : \|b\|_{*,w} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{w(B(x,r))} \int_{B(x,r)} |b(y) - b_{B(x,r),w}| w(y) dy < \infty\}$$

and

$$b_{B(x,r),w} = \frac{1}{w(B(x,r))} \int_{B(x,r)} b(y) w(y) dy.$$

Remark 7. (1) The John-Nirenberg inequality : there are constants $C_1, C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in B : |b(x) - b_B| > \beta\}| \leq C_1 |B| e^{-C_2 \beta / \|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) For $1 < p < \infty$ the John-Nirenberg inequality implies that

$$(4.1) \quad \|b\|_* \approx \sup_B \left(\frac{1}{|B|} \int_B |b(y) - b_B|^p dy \right)^{\frac{1}{p}}$$

and for $1 \leq p < \infty$ and $w \in A_\infty$

$$(4.2) \quad \|b\|_* \approx \sup_B \left(\frac{1}{w(B)} \int_B |b(y) - b_B|^p w(y) dy \right)^{\frac{1}{p}}.$$

Indeed, from the John-Nirenberg inequality and using Lemma 2 (3), we get

$$w(\{x \in B : |b(x) - b_B| > \beta\}) \leq C w(B) e^{-C_2 \beta \delta / \|b\|_*}$$

for some $\delta > 0$. Hence, this inequality implies that

$$\begin{aligned} \int_B |b(y) - b_B|^p w(y) dy &= p \int_0^\infty \beta^{p-1} w(\{x \in B : |b(x) - b_B| > \beta\}) d\beta \\ &\leq C w(B) \int_0^\infty \beta^{p-1} e^{-C_2 \beta \delta / \|b\|_*} d\beta \\ &= C w(B) \|b\|_*^p. \end{aligned}$$

To prove that required equivalence we also need to have the right hand inequality, which is easily obtained using the Hölder's inequality, then we get (4.2). Note that (4.1) follows from (4.2) in the case of $w \equiv 1$.

(3) Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$(4.3) \quad |b_{B(x,r)} - b_{B(x,t)}| \leq C \|b\|_* \ln \frac{t}{r} \text{ for } 0 < 2r < t,$$

where C is independent of b, x, r and t .

Lemma 5. [[14], Proposition 7.1.2] (see also [[29], Theorem 5]) Let $w \in A_\infty$ and $1 < p < \infty$. Then the following statements are equivalent:

$$(1) \quad \|b\|_* \approx \sup_B \left(\frac{1}{|B|} \int_B |b(y) - b_B|^p dy \right)^{\frac{1}{p}},$$

$$(2) \quad \|b\|_* \approx \sup_B \inf_{a \in \mathbb{R}} \frac{1}{|B|} \int_B |b(y) - a| dy,$$

$$(3) \quad \|b\|_{*,w} = \sup_B \frac{1}{w(B)} \int_B |b(y) - b_{B,w}| w(y) dy.$$

The following lemma has been proved in [20].

Lemma 6. (see [20]) *The following statements hold.*

i) *Let $w \in A_\infty$ and b be a function in $BMO(\mathbb{R}^n)$. Let also $1 \leq p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{w(B(x, r_1))} \int_{B(x, r_1)} |b(y) - b_{B(x, r_2), w}|^p w(y) dy \right)^{\frac{1}{p}} \leq C[w]_{A_\infty}^{2^n} \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_*,$$

where $C > 0$ is independent of f , w , x , r_1 and r_2 .

ii) *Let $w \in A_p$ and b be a function in $BMO(\mathbb{R}^n)$. Let also $1 < p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{w^{1-p'}(B(x, r_1))} \int_{B(x, r_1)} |b(y) - b_{B(x, r_2), w}|^{p'} w(y)^{1-p'} dy \right)^{\frac{1}{p'}} \leq C[w]_{A_p}^{\frac{1}{p}} \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_*,$$

where $C > 0$ is independent of f , w , x , r_1 and r_2 .

From Lemma 6 we get the following corollary, which has been proved in [22] for $w \equiv 1$.

Corollary 6. *Let b be a function in $BMO(\mathbb{R}^n)$. Let also $1 \leq p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x, r_1)|} \int_{B(x, r_1)} |b(y) - b_{B(x, r_2)}|^p dy \right)^{\frac{1}{p}} \leq C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_*,$$

where $C > 0$ is independent of f , x , r_1 and r_2 .

Theorem 4. (see [25]) *Let $\Omega \in L_s(S^{n-1})$, $s > 1$, be homogeneous of degree zero, and $1 < p < \infty$. If $b \in BMO(\mathbb{R}^n)$ and p , q , w satisfy one of the following conditions, then $[b, \overline{T}_\Omega]$ is bounded on $L_p(w)$:*

- (i) $s' \leq p < \infty$, $p \neq 1$ and $w \in A_{\frac{p}{s'}}_p$;
- (ii) $1 < p \leq s$, $p \neq \infty$ and $w^{1-p'} \in A_{\frac{p'}{s'}}_p$;
- (iii) $1 < p < \infty$, and $w^{s'} \in A_p$.

As in the proof of Theorem 3, it suffices to prove the following main Lemma 7.

Lemma 7. (Our main Lemma) *Let $\Omega \in L_s(S^{n-1})$, $s > 1$, be homogeneous of degree zero. Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$, and $T_{\Omega, b}$ is a sublinear operator satisfying condition (1.2), bounded on $L_p(w)$. Then, for $s' \leq p$ and $w \in A_{\frac{p}{s'}}_p$ the*

inequality
(4.4)

$$\|T_{\Omega,b}f\|_{L_{p,w}(B(x_0,r))} \lesssim \|b\|_* w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0, r)$ and for all $f \in L_{p,w}^{loc}(\mathbb{R}^n)$.

Also, for $p < s$ and $w^{1-p'} \in A_{\frac{p'}{s-p}}^{p'}$ the inequality

$$\|T_{\Omega,b}f\|_{L_{p,w}(B(x_0,r))} \lesssim \|b\|_* \|w\|_{L_{\frac{s}{s-p}}(B(x_0,r))}^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} \|w\|_{L_{\frac{s}{s-p}}(B(x_0,t))}^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0, r)$ and for all $f \in L_{p,w}^{loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. As in the proof of Lemma 3, we represent function f in form (3.4) and have

$$\|T_{\Omega,b}f\|_{L_{p,w}(B)} \leq \|T_{\Omega,b}f_1\|_{L_{p,w}(B)} + \|T_{\Omega,b}f_2\|_{L_{p,w}(B)}.$$

For $s' \leq p$ and $w \in A_{\frac{p}{s'}}^{p'}$, from the boundedness of $T_{\Omega,b}$ on $L_p(w)$ (see Theorem 4) it follows that:

$$\begin{aligned} \|T_{\Omega,b}f_1\|_{L_{p,w}(B)} &\leq \|T_{\Omega,b}f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &\lesssim \|b\|_* \|f_1\|_{L_{p,w}(\mathbb{R}^n)} = \|b\|_* \|f\|_{L_{p,w}(2B)}. \end{aligned}$$

It is known that $x \in B$, $y \in (2B)^C$, which implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. Then for $x \in B$, we have

$$\begin{aligned} |T_{\Omega,b}f_2(x)| &\lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |b(y) - b(x)| |f(y)| dy \\ &\approx \int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^n} |b(y) - b(x)| |f(y)| dy. \end{aligned}$$

Hence, we get

$$\begin{aligned} \|T_{\Omega,b}f_2\|_{L_{p,w}(B)} &\lesssim \left(\int_B \left(\int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^n} |b(y) - b(x)| |f(y)| dy \right)^p w(x) dx \right)^{\frac{1}{p}} \\ &\lesssim \left(\int_B \left(\int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^n} |b(y) - b_{B,w}| |f(y)| dy \right)^p w(x) dx \right)^{\frac{1}{p}} \\ &\quad + \left(\int_B \left(\int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^n} |b(x) - b_{B,w}| |f(y)| dy \right)^p w(x) dx \right)^{\frac{1}{p}} \\ &= J_1 + J_2. \end{aligned}$$

We have the following estimation of J_1 . When $s' \leq p$, by the Fubini's theorem

$$\begin{aligned}
J_1 &\approx w(B(x_0, r))^{\frac{1}{p}} \int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^n} |b(y) - b_{B,w}| |f(y)| dy \\
&\approx w(B(x_0, r))^{\frac{1}{p}} \int_{(2B)^C} |\Omega(x-y)| |b(y) - b_{B,w}| |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\
&\approx w(B(x_0, r))^{\frac{1}{p}} \int_{2r}^{\infty} \int_{2r \leq |x_0-y| \leq t} |\Omega(x-y)| |b(y) - b_{B,w}| |f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim w(B(x_0, r))^{\frac{1}{p}} \int_{2r}^{\infty} \int_{B(x_0, t)} |\Omega(x-y)| |b(y) - b_{B,w}| |f(y)| dy \frac{dt}{t^{n+1}} \text{ holds.}
\end{aligned}$$

Applying the Hölder's inequality and by Lemma 6, (3.3) and (2.4), we get

$$\begin{aligned}
J_1 &\lesssim \|b\|_* w(B(x_0, r))^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|\Omega(\cdot - y)\|_{L_s(B(x_0, t))} \|f\|_{L_{s'}(B(x_0, t))} \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_* w(B(x_0, r))^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0, t))} \left\|w^{-\frac{s'}{p}}\right\|_{L_{(\frac{p}{s'})'}(B(x_0, t))}^{\frac{1}{s'}} |B(x_0, 2t)|^{\frac{1}{s}} \frac{dt}{t^{n+1}} \\
&\lesssim \|b\|_* w(B(x_0, r))^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0, t))} w(B(x_0, t))^{-\frac{1}{p}} \frac{dt}{t}.
\end{aligned}$$

In order to estimate J_2 note that

$$J_2 = \|(b(\cdot) - b_{B(x_0, t), w})\|_{L_{p,w}(B(x_0, t))} \int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^n} |f(y)| dy.$$

By (3.6) and Lemma 6, we get

$$\begin{aligned}
J_2 &\lesssim \|b\|_* w(B(x_0, r))^{\frac{1}{p}} \int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^n} |f(y)| dy \\
&\lesssim \|b\|_* w(B(x_0, r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0, t))} w(B(x_0, t))^{-\frac{1}{p}} \frac{dt}{t}.
\end{aligned}$$

Summing up J_1 and J_2 , for all $p \in (1, \infty)$ we get

$$\|T_{\Omega, b} f_2\|_{L_{p,w}(B)} \lesssim \|b\|_* w(B(x_0, r))^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0, t))} w(B(x_0, t))^{-\frac{1}{p}} \frac{dt}{t}$$

Finally, we have the following

$$\|T_{\Omega, b} f\|_{L_{p,w}(B)} \lesssim \|b\|_* \|f\|_{L_{p,w}(2B)} + \|b\|_* w(B(x_0, r))^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0, t))} w(B(x_0, t))^{-\frac{1}{p}} \frac{dt}{t}.$$

On the other hand by (3.8), we have

$$\|T_{\Omega,b}f\|_{L_{p,w}(B)} \lesssim \|b\|_* w(B(x_0, r))^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0, t))} w(B(x_0, t))^{-\frac{1}{p}} \frac{dt}{t}.$$

For the case of $1 < p < s$, $w^{1-p'} \in A_{\frac{p'}{s'}}$, we can also use the same method, so we omit the details. This completes the proof of Lemma 7. \square

Now we can give the following theorem (our main result).

Theorem 5. (Our main result) Suppose that $\Omega \in L_s(S^{n-1})$, $s > 1$, is homogeneous of degree zero and $T_{\Omega,b}$ is a sublinear operator satisfying condition (1.2), bounded on $L_p(w)$. Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$.

Let also, for $s' \leq p$ and $w \in A_{\frac{p}{s'}}$ the pair (φ_1, φ_2) satisfies the condition

$$(4.5) \quad \int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{essinf}_{t < \tau < \infty} \varphi_1(x, \tau) w(B(x, \tau))^{\frac{1}{p}}}{w(B(x, t))^{\frac{1}{p}}} \frac{dt}{t} \leq C \varphi_2(x, r),$$

and for $p < s$ and $w^{1-p'} \in A_{\frac{p'}{s'}}$ the pair (φ_1, φ_2) satisfies the condition

$$(4.6) \quad \int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{essinf}_{t < \tau < \infty} \varphi_1(x, \tau) \|w\|_{L_{\frac{s}{s-p}}(B(x, r))}^{\frac{1}{p}}}{\|w\|_{L_{\frac{s}{s-p}}(B(x, t))}^{\frac{1}{p}}} \frac{dt}{t} \leq C \varphi_2(x, r) \frac{w(B(x, r))^{\frac{1}{p}}}{\|w\|_{L_{\frac{s}{s-p}}(B(x, r))}^{\frac{1}{p}}},$$

where C does not depend on x and r .

Then, the operator $T_{\Omega,b}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover

$$(4.7) \quad \|T_{\Omega,b}f\|_{M_{p,\varphi_2}(w)} \lesssim \|b\|_* \|f\|_{M_{p,\varphi_1}(w)}.$$

Proof. since $f \in M_{p,\varphi_1}(w)$, by (2.5) and the non-decreasing, with respect to t , of the norm $\|f\|_{L_{p,w}(B(x_0, t))}$, we get

$$\begin{aligned} & \frac{\|f\|_{L_{p,w}(B(x_0, t))}}{\operatorname{essinf}_{0 < t < \tau < \infty} \varphi_1(x_0, \tau) w(B(x_0, \tau))^{\frac{1}{p}}} \\ & \leq \operatorname{esssup}_{0 < t < \tau < \infty} \frac{\|f\|_{L_{p,w}(B(x_0, t))}}{\varphi_1(x_0, \tau) w(B(x_0, \tau))^{\frac{1}{p}}} \\ & \leq \operatorname{esssup}_{0 < \tau < \infty} \frac{\|f\|_{L_{p,w}(B(x_0, \tau))}}{\varphi_1(x_0, \tau) w(B(x_0, \tau))^{\frac{1}{p}}} \\ & \leq \|f\|_{M_{p,\varphi_1}(w)}. \end{aligned}$$

For $s' \leq p < \infty$, since (φ_1, φ_2) satisfies (4.5), we have

$$\begin{aligned}
& \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t} \\
& \leq \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L_{p,w}(B(x_0,t))}}{\operatorname{ess\,inf}_{t < \tau < \infty} \varphi_1(x_0, \tau) w(B(x_0, \tau))^{\frac{1}{p}}} \frac{\operatorname{ess\,inf}_{t < \tau < \infty} \varphi_1(x_0, \tau) w(B(x_0, \tau))^{\frac{1}{p}}}{w(B(x_0, t))^{\frac{1}{p}}} \frac{dt}{t} \\
& \leq C \|f\|_{M_{p, \varphi_1}(w)} \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t < \tau < \infty} \varphi_1(x_0, \tau) w(B(x_0, \tau))^{\frac{1}{p}}}{w(B(x_0, t))^{\frac{1}{p}}} \frac{dt}{t} \\
& \leq C \|f\|_{M_{p, \varphi_1}(w)} \varphi_2(x_0, r).
\end{aligned}$$

Then by (4.4), we get

$$\begin{aligned}
\|T_{\Omega, b} f\|_{M_{p, \varphi_2}(w)} &= \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_2(x_0, r)^{-1} w(B(x_0, r))^{-\frac{1}{p}} \|T_{\Omega, b} f\|_{L_{p,w}(B(x_0, r))} \\
&\leq C \|b\|_* \sup_{x_0 \in \mathbb{R}^n, r > 0} \varphi_2(x_0, r)^{-1} \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t} \\
&\leq C \|b\|_* \|f\|_{M_{p, \varphi_1}(w)}.
\end{aligned}$$

For the case of $1 < p < s$, we can also use the same method, so we omit the details. This completes the proof of Theorem 5. \square

For the sublinear commutator of the fractional maximal operator with rough kernel which is defined as follows

$$M_{\Omega, b}(f)(x) = \sup_{t > 0} |B(x, t)|^{-1} \int_{B(x, t)} |b(x) - b(y)| |\Omega(x - y)| |f(y)| dy$$

and for the linear commutator of the singular integral $[b, \overline{T}_\Omega]$ by Theorem 5, we get the following new result.

Corollary 7. *Suppose that $\Omega \in L_s(S^{n-1})$, $s > 1$, is homogeneous of degree zero, $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. If for $s' \leq p$ and $w \in A_{\frac{p}{s'}}$ the pair (φ_1, φ_2) satisfies condition (4.5) and for $p < s$ and $w^{1-p'} \in A_{\frac{p'}{s'}}$ the pair (φ_1, φ_2) satisfies condition (4.6). Then, the operators $M_{\Omega, b}$ and $[b, \overline{T}_\Omega]$ are bounded from $M_{p, \varphi_1}(w)$ to $M_{p, \varphi_2}(w)$.*

In the case of $w = 1$ from Theorem 5, we get

Corollary 8. *(see [2, 17]) Suppose that $\Omega \in L_s(S^{n-1})$, $s > 1$, is homogeneous of degree zero and $T_{\Omega, b}$ is a sublinear operator satisfying condition (1.2), bounded on $L_p(\mathbb{R}^n)$. Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Let also, for $s' \leq p$ the pair (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t < \tau < \infty} \varphi_1(x, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \leq C \varphi_2(x, r),$$

and for $p < s$ the pair (φ_1, φ_2) satisfies the condition

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t < \tau < \infty} \varphi_1(x, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{p} - \frac{n}{s} + 1}} dt \leq C \varphi_2(x, r) r^{\frac{n}{s}},$$

where C does not depend on x and r .

Then, the operator $T_{\Omega, b}$ is bounded from M_{p, φ_1} to M_{p, φ_2} . Moreover

$$\|T_{\Omega, b} f\|_{M_{p, \varphi_2}} \lesssim \|b\|_* \|f\|_{M_{p, \varphi_1}}.$$

In the case of $\varphi_1(x, r) = \varphi_2(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$ from Theorem 5, we get the following new result.

Corollary 9. Let $1 < p < \infty$, $\Omega \in L_s(S^{n-1})$, $s > 1$, be homogeneous of degree zero, $0 < \kappa < 1$ and $b \in BMO(\mathbb{R}^n)$. Let also $T_{\Omega, b}$ be a sublinear operator satisfying condition (1.2) and bounded on $L_p(w)$. For $s' \leq p$ and $w \in A_{\frac{p}{s'}}$ or $p < s$ and $w^{1-p'} \in A_{\frac{p'}{s'}}$, the operator $T_{\Omega, b}$ is bounded on the weighted Morrey spaces $L_{p, \kappa}(w)$.

When $\Omega \equiv 1$, from Theorem 5 we get

Corollary 10. Let $1 < p < \infty$, $w \in A_p$, $b \in BMO(\mathbb{R}^n)$ and the pair (φ_1, φ_2) satisfies condition (4.5). Let also T_b be a sublinear operator satisfying condition (1.2) and bounded on $L_p(w)$. Then the operator T_b is bounded from $M_{p, \varphi_1}(w)$ to $M_{p, \varphi_2}(w)$.

Remark 8. Corollary 10 has been proved in [20].

When $\Omega \equiv 1$, in the case of $\varphi_1(x, r) = \varphi_2(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$, from Theorem 5 we get the following new result.

Corollary 11. $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$ and $b \in BMO(\mathbb{R}^n)$. Let also T_b be a sublinear operator satisfying condition (1.2) and bounded on $L_p(w)$. Then the operator T_b is bounded on the weighted Morrey spaces $L_{p, \kappa}(w)$.

Remark 9. Note that, from Corollary 11 for the operators M_b and $[b, \overline{T}]$ we get results which are proved in [21].

Conclusion 1. Let $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$ and $b \in BMO(\mathbb{R}^n)$. Then, the operators M_b and $[b, \overline{T}]$ are bounded on the weighted Morrey spaces $L_{p, \kappa}(w)$.

Now, we give the applications of Theorem 3 and Theorem 5 for the Marcinkiewicz operator.

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in \mathbb{R}^n equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(\mu x) = \Omega(x), \text{ for any } \mu > 0, x \in \mathbb{R}^n \setminus \{0\}.$$

(b) Ω has mean zero on S^{n-1} , that is,

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0,$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$.

(c) $\Omega \in Lip_\gamma(S^{n-1})$, $0 < \gamma \leq 1$, that is there exists a constant $M > 0$ such that,

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma \text{ for any } x', y' \in S^{n-1}.$$

In 1958, Stein [38] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

Since Stein's work in 1958, the continuity of Marcinkiewicz integral has been extensively studied as a research topic and also provides useful tools in harmonic analysis [25, 39, 40, 41].

The sublinear commutator of the operator μ_Ω is defined by

$$[b, \mu_\Omega](f)(x) = \left(\int_0^\infty |F_{\Omega,t,b}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t,b}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y) dy.$$

We consider the space $H = \{h : \|h\| = \left(\int_0^\infty |h(t)|^2 \frac{dt}{t^3} \right)^{1/2} < \infty\}$. Then, it is clear that $\mu_\Omega(f)(x) = \|F_{\Omega,t}(x)\|$.

By the Minkowski inequality and the conditions on Ω , we get

$$\mu_\Omega(f)(x) \leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \leq C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy.$$

Thus, μ_Ω satisfies the condition (1.1). It is known that μ_Ω and $[b, \mu_\Omega]$ are bounded on $L_p(w)$ for $s' \leq p$ and $w \in A_{\frac{p}{s'}}^p$ or for $1 < p < s$ and $w^{1-p'} \in A_{\frac{p'}{s'}}^p$ (see [9, 10]), then from Theorems 3 and 5 we get

Corollary 12. *Let $\Omega \in L_s(S^{n-1})$, $s > 1$, $1 \leq p < \infty$. Let also, for $s' \leq p$ and $w \in A_{\frac{p}{s'}}^p$ the pair (φ_1, φ_2) satisfies condition (3.11) and for $p < s$ and $w^{1-p'} \in A_{\frac{p'}{s'}}^p$ the pair (φ_1, φ_2) satisfies condition (3.12) and Ω satisfies conditions (a)–(c). Then the operator μ_Ω is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$ for $p = 1$.*

Corollary 13. *Let $\Omega \in L_s(S^{n-1})$, $s > 1$, $1 \leq p < \infty$, $0 < \kappa < 1$ and $w \in A_{\frac{p}{s'}}^p$ or $w^{1-p'} \in A_{\frac{p'}{s'}}^p$. Suppose that Ω satisfies conditions (a)–(c). Then the operator μ_Ω is bounded on $L_{p,\kappa}(w)$ for $p > 1$ and bounded from $L_{1,\kappa}(w)$ to $WL_{1,\kappa}(w)$.*

Corollary 14. *Let $\Omega \in L_s(S^{n-1})$, $s > 1$. Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Let also, for $s' \leq p$ and $w \in A_{\frac{p}{s'}}$ the pair (φ_1, φ_2) satisfies condition (4.5) and for $p < s$ and $w^{1-p'} \in A_{\frac{p'}{s'}}$ the pair (φ_1, φ_2) satisfies condition (4.6) and Ω satisfies conditions (a)–(c). Then, the operator $[b, \mu_\Omega]$ is bounded from $M_{p, \varphi_1}(w)$ to $M_{p, \varphi_2}(w)$.*

Corollary 15. *Let $\Omega \in L_s(S^{n-1})$, $s > 1$, $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_{\frac{p}{s}}$ or $w^{1-p'} \in A_{\frac{p'}{s'}}$ and $b \in BMO(\mathbb{R}^n)$. Suppose that Ω satisfies conditions (a)–(c). Then the operator $[b, \mu_\Omega]$ is bounded on $L_{p, \kappa}(w)$.*

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